Tutorial 1

Q1 Ray topology

Consider the following "topology" on $\mathbb{R}$

$$\mathcal{T}_r = \{ (a,\infty) \mid a \in \mathbb{R} \} \cup \{\emptyset\} \cup \{\mathbb{R}\}$$

Prove this is a topology and give a basis

(1) $\emptyset \in \mathcal{T}_r$ $\quad \mathbb{R} \in \mathcal{T}_r$

(2) Arbitrary Unions of rays are rays

$R_a = (a, \infty)$

Let $I \subseteq \mathbb{R}$

Is $\bigcup\limits_{a \in I} R_a$ a ray?

Yes $\bigcup\limits_{a \in I} R_a = (\inf I, \infty)$ or $\mathbb{R}$ if inf does not exist

(3) Finite intersection $J \subseteq \mathbb{R}$

$\bigcap\limits_{a \in J} R_a = (\max(J), \infty)$ because $|J| < \infty$

Part 2

$X = \mathbb{R}$, $\mathcal{J}$ = countable sets $\mathbb{R} \cup \{\mathbb{R}\}$

$I = [0,1]$

$U_i = \{i\}$ is open

$\bigcup_{i \in I} U_i = \bigcup_{i \in I} \{i\} = [0,1] \in \mathcal{J}$

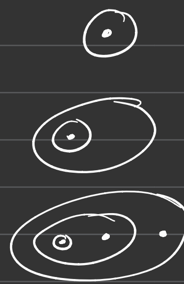
X with cardinality μ subsets with cardinality $< \mu$ do not form a topology

Q2

Construct a topology with n open sets.

Hint: Find one with n and construct one with $n+1$

	X	$\mathcal{T}$	
$n=1$	$\emptyset$	$\mathcal{T} = \{\emptyset\}$	X_0
$n=2$	$\{p\}$	$\mathcal{T} = \{\emptyset, X\}$	X_1
$n=3$	$\{p_1, p_2\}$		X_2
$n=4$	$\{p_1, p_2, p_3\}$		X_3



How to construct X_{n+1} from X_n

say $X_n = \{p_1, \ldots, p_{n-1}\}$

(X_n = topology with n open sets)

Let $\mathcal{T}_n$ be the topology with n open sets

To construct X_{n+1} and $\mathcal{T}_{n+1}$, Let

$$X_{n+1} = X_n \cup \{*\}$$

$$\mathcal{J}_{n+1} = \mathcal{J}_n \cup X_{n+1}$$

Why is this a topology

(1) $\emptyset, X_{n+1} \in \mathcal{J}_{n+1}$

(2) Finite unions: if the finite union includes X_{n+1}

$\Rightarrow$ union is $X_{n+1} \in \mathcal{J}_{n+1}$

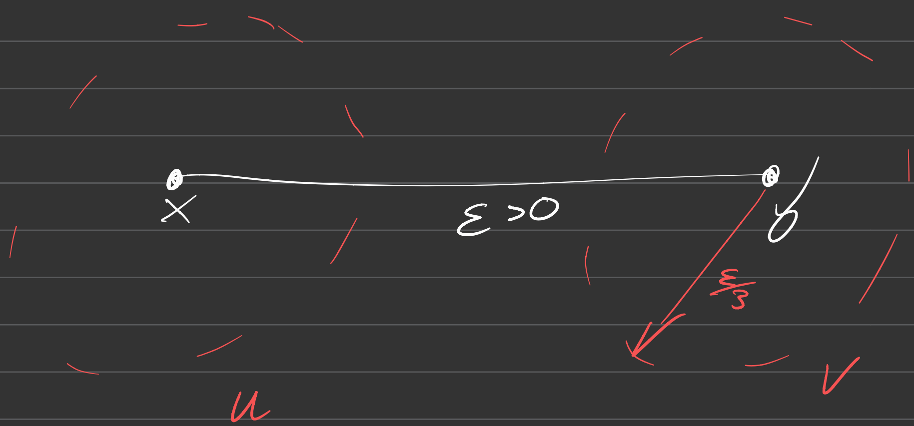
If not induction

(3) Similarly for finite intersections, if the intersection includes X_{n+1}
$\Rightarrow$ can remove X_{n+1} from intersection. If not, induction

Q3

Let $x \neq y \in (X, d)$

Show ~~that~~ ~~there~~ exists $U \ni x$, $V \ni y$
st $U \cap V = \emptyset$ and U, V open



$U = B_x\left(\frac{\varepsilon}{3}\right) = B_{\frac{\varepsilon}{3}}(x)$

$V = B_y\left(\frac{\varepsilon}{3}\right)$

U, V are basis elements

($\Rightarrow$ open)

Only have to show

$x \in B_x(\frac{\varepsilon}{3})$ $\quad d(x,x) = 0 < \frac{\varepsilon}{3}$

$y \in B_y(\frac{\varepsilon}{3})$ $\quad \ldots -$

$U \cap V =$ empty. Assume not. Let $z \in U \cap V$

$d(x,z) < \frac{\varepsilon}{3}, \quad d(z,y) < \frac{\varepsilon}{3}$

$\varepsilon < d(x,y) \leq d(x,z) + d(y,z) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$

$\Rightarrow$ But by assumption $\varepsilon = d(x,y)$
so impossible

Hausdorff

$x \neq y$: be able to separate x, y
by open sets

"Separation Condition"

Hausdorff $\Leftrightarrow \forall\, x, y \in X,\ x \neq y,\ \exists\, u, v$

st $x \in U$, $y \in V$, U, V open $U \cap V = \emptyset$

3(ii)

Particular point not metrisable
(unless $|X| = 1$)

$x \neq y$, open sets $x \in U$, $y \in V$

It is not possible that $U \cap V = \emptyset$?

No $p \in U$, $p \in V$

$\Rightarrow p \in U \cap V \Rightarrow$ not Hausdorf

$\Rightarrow$ not metrisable

<u>Q4</u>

Finite top space

<u>Fact</u> Finite top space

metrisable $\Leftrightarrow$ Hausdorf

$\Updownarrow$ $\qquad$ $\Leftrightarrow$

topology = powerset

$\Leftrightarrow$

Singletons open

p.2H $p \in X$

$\{p\}$ is open

HD $\Rightarrow$ $\forall$ $x \in X$, $x \neq p$ $\exists$ U_x, V_x st

$p \in U_x$, $x \in V_x$ and $U_x \cap V_x = \emptyset$

$\bigcap_{x \in X} U_x = \{p\}$

$|X| < \infty$

finite intersection of open sets

HW 2

Q1 (a) $\overline{(0,1)} = [0,1]$

$(0,1) \subseteq \overline{(0,1)}$

What are the limit points

$x<0 \Rightarrow x$ is not a limit point

$|x-0| = |x|$

$$\underbrace{\left(x - \frac{|x|}{2}, x + \frac{|x|}{2}\right)} \cap (0,1) = \emptyset$$

$x \in$ this set, set given $\Rightarrow x \notin (0,1)$

$x \in S' \Leftrightarrow$ every nbhd of x intersects S in a point ~~other~~ then x

$\delta \in (-\delta, \delta) \qquad (\delta > 0,\ \delta < 1)$

$(-\delta, \delta) \cap (0,1) = (0, \delta) \neq \emptyset$

$\Rightarrow (-\delta, \delta)$ intersects $(0,1)$ in a point other than 0

$\Rightarrow 0 \in (0,1)' \Rightarrow 0 \in \overline{(0,1)}$

$\Rightarrow \{0,1\} = (0,1)' \Rightarrow \overline{(0,1)} = [0,1]$

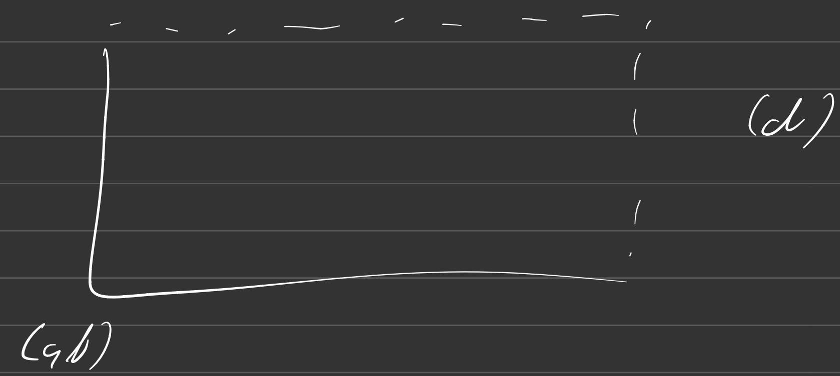
(d) $\overline{\{1,3,5,\dots\}}$

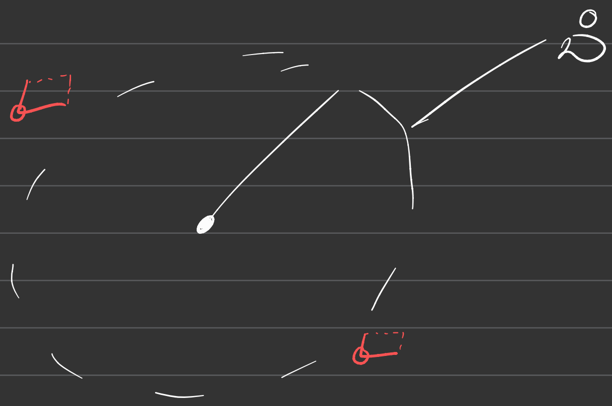
All closed sets in the finite complement ~~topology~~ are <u>either</u> finite or $\mathbb{R}$

$\Rightarrow \{1,3,\dots\} \subseteq \overline{\{1,3,\dots\}} \Rightarrow \overline{\{1,3,5,\dots\}} = \mathbb{R}$

(d) $\mathbb{R}_\ell$ $\quad [a,b)$ $\quad$ basis elements for $\mathbb{R}_\ell$

$[a,b) \times [c,d)$ ———— $\mathbb{R}_\ell \times \mathbb{R}_\ell$



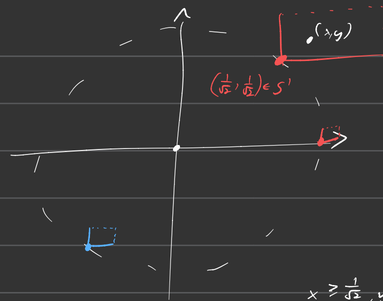


(1) $\mathring{D} \subseteq \overline{\mathring{D}}$, $\quad \mathring{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

$\mathring{D} \cup S^1 = D$

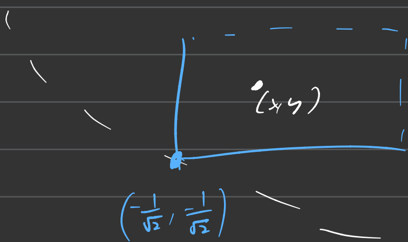
• $x \notin D \Rightarrow x \notin \overset{\circ}{\overline{D}}$



$x \geq \frac{1}{\sqrt{2}}, y \geq \frac{1}{\sqrt{2}}$

$x^2 + y^2 \geq \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$

$\Rightarrow (x,y) \notin \overset{\circ}{D}$



$x \in \left(-\frac{1}{\sqrt{2}}, 0\right)$ $\sim$ $x \in (-t, 0)$

$y \in \left(-\frac{1}{\sqrt{2}}, 0\right)$ $\quad$ $y \in \left(-\sqrt{1-t^2}, 0\right)$

$x^2 + y^2 < 1$

Q2 X is Hausdorff $\Leftrightarrow$ $\Delta \subseteq X \times X$ is closed

$\forall x \neq y$, $\exists U \ni x$, $V \ni y$ st U, V open $U \cap V = \emptyset$

Do ~~not~~ show that $X \times X$ is hausdorff
show that X is hausdorff

Everyone did this well enough according to Ian

Q3 $|x| : \mathbb{R}_r \to \mathbb{R}_r$
$= f$

f is not continuous

To show this, use definitions

f cts $\Leftrightarrow$ $\forall$ $U \subseteq \mathbb{R}_r$ open
$f^{-1}(U)$ is open

find U open st $f^{-1}(U)$ is not open

$U = (1, \infty)$

Warning!

$\mathbb{R}_r$ with basis (a, ∞)

the other one $(-\infty, a)$

$f^{-1}((a,\infty))$

$= (-\infty,-1) \cup (1,\infty)$

$= (-\infty,-1) \cup (1,\infty)$ not a ray in this topology

$g = e^x : \mathbb{R}_r \to \mathbb{R}_r$

$g^{-1}(a,\infty) = \begin{cases} \mathbb{R} & a \le 0 \\ (\ln(a),\infty) & \end{cases}$ are open!

$\Rightarrow g$ cts

Q4

(b) 2nd countable : has countable basis

1st countable : has countable nbhd basis ie

$\forall\, x \in X,\ \exists \{U_1^x, U_2^x, \dots\}$

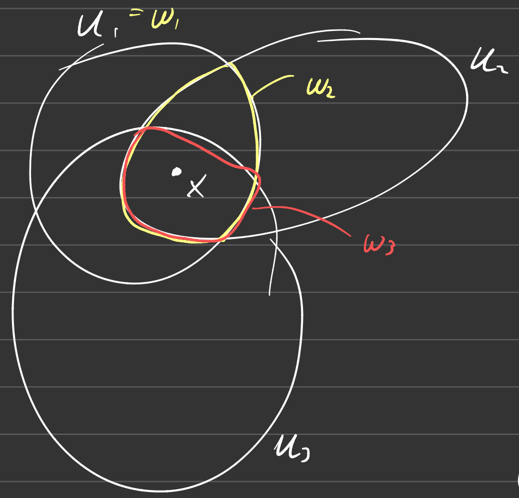
st $\forall$ nbhd U of x

$\exists\, U_i^x \subseteq U_i$

$x \in \bar{A}$, $A \subset X$, X 1st countable

$\Rightarrow \exists (x_n) \subset A$ st $x_n \longrightarrow x$

$W_1 = U_1$, $W_2 = U_1 \cap U_2$ $\quad \{U_i\}$ is a nbhd basis at x



$x_1 \in W_1 \cap A$

$x_2 \in W_2 \cap A$

$x_3 \in W_3 \cap A$

(needs choice)

Tutorial 3

$$\begin{tikzcd} D^2 \arrow[d, "q"'] \arrow[dr] & \\ D^2/\sim & S^2 \end{tikzcd}$$

$D^2 \subseteq \mathbb{R}^2$

D^2 compact

$[0,1]$ compact

$[-1,1] \times \underbrace{[-1,1]}_{I}$ compact

$D^2 \subseteq I^2$

D^2 closed $\Rightarrow$ D^2 closed

S^2 HD

$S^2 \subseteq \mathbb{R}^3$ HD

Use spherical and polar co-ords

$S^2 = \{(\theta, \varphi, r) \in \mathbb{R}^3 \mid r = 1\}$

$D^2 = \{(\varphi, r)\}$

$$\begin{tikzcd}
D^2 \arrow[d, "p"'] \arrow[dr, "g"] & \\
D^2/\sim \arrow[r, "f"'] & S^2
\end{tikzcd}$$

$\overbrace{\text{closed surj cts}}^{\text{quotient}}$

f is a quotient map

$D^2 \xrightarrow{g} S^2$

$(\varphi, r) \longrightarrow (\varphi, \theta, r)$

$(\varphi, r\pi, r)$

$(\varphi, r) \xrightarrow{g} (\varphi, r\pi, 1)$

if $r = 1$

$\Rightarrow g(\varphi, 1) = (\varphi, \pi, 1)$

$=$ south pole

$\partial D^2 = S^1$ is mapped to S^1

otherwise if $r \neq 1$ then

$g(\varphi, r) = g(\varphi', r')$

$\Rightarrow \varphi = \varphi', \; r = r'$

The fibres of p are ~~either~~

- (r,θ) $r<1$
- or $S^1 = \{r=1\}$

g is constant on ~~the~~ fibres

$\Rightarrow$ ~~induces~~ map f

D^2 compact S^2 HD

$g: D^2 \longrightarrow S^2$ is a closed map

$\Rightarrow$ g is a quotient map

$\Rightarrow$ f is a quotient map

f injective $\Rightarrow$ h

$f([r]) = f([r'])$

$\Rightarrow [r] = [r']$

$\pi : X \times Y \longrightarrow X$ is open

basis $U \times V$

$\overline{\pi}(U \times V) = U$ open V

$U = \bigcup_{\alpha \in I} B_\alpha$, B_α basis elements

$\overline{\pi}(U) = \overline{\pi}\left(\bigcup_{\alpha \in I} B_\alpha\right) = \bigcup_{\alpha \in I} \overline{\pi}(B_\alpha)$

In general

$f\left(\bigcup U_\alpha\right)$

$= \bigcup f(U_\alpha)$

and $\overline{\pi}(U \times V) = U$

basis element

open set

HW 4

(1) Deleted ray topology

$\mathcal{T} = \{(-\infty, a) \setminus K \mid K = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$

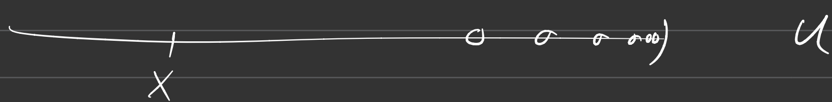
To show: connected

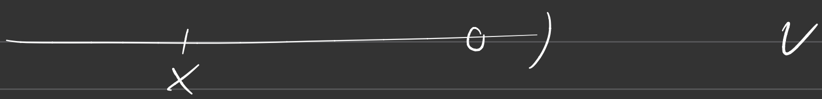
Suppose

$\mathbb{R} = U \cup V$, a separation

Then $U = (-\infty, a) \setminus K$

$V = (-\infty, b) \setminus K$





$x = \min(0, a, b) - 2$

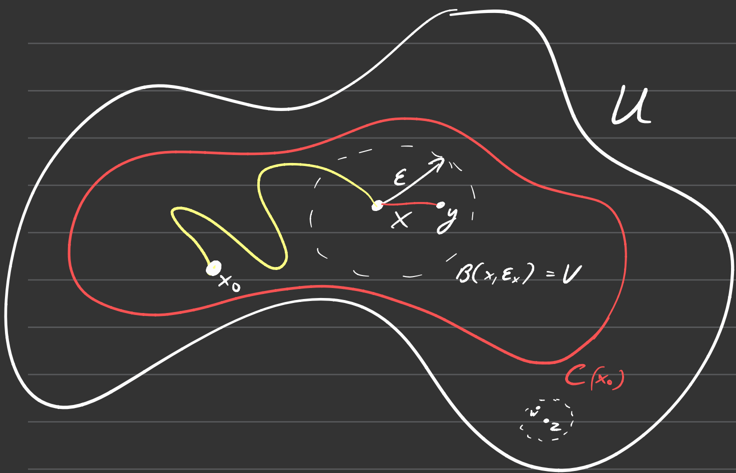
$\Rightarrow x \in U \cap V \Rightarrow U \cap V \Rightarrow U \cap V$ is not a separation

Q2 Open connected subset of $\mathbb{R}^2$ is path connected

Let $x_0 \in U$ open, connected

$C(x_0) = \{x \in U \mid \exists$ path γ from x_0 to $x\}$

Show that $C(x_0)$ is both open and closed



(1) $C(x_0)$ is open

Let $x \in C(x_0)$. To show there exist some open set $x \in V \subseteq C(x_0)$

Since the path $x_0 \to x$ and the path $x \to y$ can be concatenated there exists a path from $x_0 \to y$ thus $y \in C(x_0) \Rightarrow B(x, \varepsilon_0) \subseteq C(x_0)$
$\Rightarrow C(x_0)$ is open

$C(x_0)$ closed

Pick a point $z \in C(x_0)^c$. Consider an open ball centred at z, say $B(z, \varepsilon_z)$. To show $B(z, \varepsilon_z) \subseteq C(x_0)^c$. If a point $w \in B(z, \varepsilon_z)$ then there exists a path γ from x_0 to w and by concatenating ~~this~~ path from w to z we get a path x_0 to z contradicting that $z \notin C(x_0)$

$x_0 \in C(x_0)$

$\Rightarrow C(x_0) \neq \emptyset$

$\Rightarrow C(x_0)$ is non empty open and closed so

$U = C(x_0) \cup C(x_0)^c$ is a separation

unless $C(x_0) = U$

$\Rightarrow$ As U is connected, $U = C(x_0)$

Q3 $[0,1]$, $(0,1]$, $(0,1)$ are all distinct ~~topologically~~

$f: X \to Y$, $Z \subseteq X$ is a homeomorphism then

$f: X \setminus Z \to Y$ is a homeomorphism also

apply to

$X = [0,1]$, $Z = \{0\}$

$[0,1] \xrightarrow[\simeq]{f} (0,1)$

$Z = \{0\}$

$\Rightarrow (0,1] \longrightarrow (0,1) - f(0)$

connected

cannot be connected

$(0, f(0)) \cup (f(0), 1)$

is a separation

Contradiction since you can't have a homeomorphism from a connected set to a disconnected set

$\Rightarrow [0,1] \not\cong (0,1)$

$z = \{0,1\}$

$\Rightarrow [0,1] \not\cong (0,1]$

$(0,1) \not\cong (0,1)$

$z = \{1\}$

Q4

Simple order ("$<$") (NOT "$\leq$")

is a linear continuum

(1) if $\exists y$ st $x < y < z$

(2) A bounded $\Rightarrow$ A has lub

Show: connected simple order is continuum

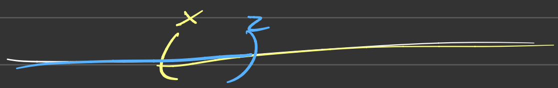
Let X be connected order topology

We want to show (1) (2), X a linear continuum

(1) Suppose $x < z$ but there exists no y st $x < y < z$

$$\Rightarrow X = (-\infty, z) \cup (x, \infty)$$

(where $-\infty$ denotes the min of X if it exists)



(2) To show least upper bounds exists